*To see a World in a Grain of Sand*
*And a Heaven in a Wild Flower,*
*Hold Innity in the palm of your hand*
*And Eternity in an hour.*

William Blake, Auguries of Innocence

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# INFINITY: ANCIENT AND NEW

Infinity is one of the strangest, richest, and most interesting notions, which the human brain ever invented. It led the ancient thinkers to many logical paradoxes, like the paradox of Akhilles and the turtle, the solution of which one had to wait about 20 centuries.

The notion has been puzzling the mankind for several thousand years. It appears in genesis legends, it is the topic of many philosophical arguments, religion and science share the notion. The ancient scientists were not specialized to one or another area, but rather they were mainly philosophers and thinkers. This way they were able to approach the notion of limitness, infinity: philosophically, astronomically, or even mathematically.

Infinity is closely related to the eternity, as the infinity of time. One of the many pictures from the ancient far East says the following: "Once every 100 years a little bird alights on the top of the diamond mountain and clears on it its nib. When the mountain completely deteriorates, then one instant is gone from the eternity."

Armenian eternity symbol, 10th century

## 1. **Greeks**

We have no written memory about the arguments on infinity from the time before the Greeks. For the Greek civilization everything what is infinite originates from the abysm (primal chaos). In their world every quantity was a finite number. For Aristotle and Pythagoras the infinity meant as odd rather than perfect. Even though Pythagoras invented un-understandable numbers like $\sqrt{2}$, but they did not have then an appropriate theory about infinity to define it properly.

**Anaximandros** (610-546 BC), the ancient Greek philosopher was the first to formulate the conjecture on the infinity of the universe, and tried to lighten the notion of unboundedness: "Independently, where the soldier stands, he can stretch out his lance a bit further yet." Or: "The principle and beginning ... of being is the limitless, where beings have their beginning, therein also have their end according to necessity; for they pay penalty and retritbution to each other for their injustice in accordance with the arrangement of time."

One can see two different approaches to infinity: Infinitely big, unbounded, and infinitely small, infinitely divisible. Time is unbounded, the space around us has no boundary, the space and time can be divided into smaller pieces without any end.

**Aristotle** (384-322 BC) is credited with being the root of a field of thought, in his influence of succeeding thinking for a period spanning more than one subsequent millennium, by his rejection of the idea of *actual infinity*. He wrote: "It is always possible to think of a larger number: for the number of times a magnitude can be bisected is infinite.

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Hence the infinite is potential, never actual; the number of parts that can be taken always surpasses any assigned number." - This is often called potential infinity. Actual infinity is for instance the collection of all natural numbers. Aristotle, and many others after him, rejected the existence of actual infinity. He wrote: "Many absurdity follows from both the negation of infinity and acceptance of infinity."

Such a big uncertainity, and unacceptence led the thinkers to avoid the notion of infinity, even in such cases when with today's view it would be obvios. **Euclid**($\sim$ 300 BC) in his book Elements shows that "there exist prime numbers more than any collection of primes", so there are infinitely many prime numbers, - but he does not state this. He also avoids the notion of infinity in geometry. He writes in the "Elements". "A point is such which has no part." This thinking avoids the usage of the notion of infinity, as this definition of a point uses the infinite divisibility of the space.

## 2. Early Indian Thinking

The **Jain Agamas** ($\sim$ 400 BC) (texts of Jainism) classifies all numbers into three sets: enumerable, innumerable, and infinite. The Jains were the first to discard the idea that all infinities were the same or equal. They recognized different types of infinities: infinite in length, in area, in volume, and infinite perpetually (infinite number of dimensions). The highest enumerable number $\mathbb{N}$ of the Jains corresponds to the modern concept of $\aleph_0$ (the cardinal number of the infinite set of integers $\{1, 2, ...\}$, the smallest cardinal transfinite number. They also defined a whole system of cardinal numbers, of which the highest enumerable number $\mathbb{N}$ is the smallest.

**For further understanding, we had to wait more than thousand years!**

## 3. Views from the Renaissance and European Scientists

**Galileo** (1564-1642) discussed the example of comparing the square numbers {1,4,9,16,...} with the natural numbers {1,2,3,4,...} as follows:

$$1 \to 1,$$

$$2 \to 4,$$

$$3 \to 9,$$

$$4 \to 16,$$

...

It appeared by this reasoning as though a set (Galileo did not use the terminology) which is naturally smaller than the set of which it is part (since it does not contain all the members) is in some sense the same size. Galileo found no way around this problem: "So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and the number of their roots in infinite; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally the attributes equal, greater, less are not applicable to infinite, but only to finite quantities."

The idea that size can be measured by one-to-one correspondence is today known as **Hume's principle**(A Treatise of Human Nature, 1740), although Hume (1711-1776), like Galileo, believed the principle could not be applied to the infinite.

**John Wallis**(1616-1703) English mathematician gave an expression for approximating $\pi$:

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}...$$

This approximation shows the necessity of applying an infinite process. In **1657** Wallis introduced the symbol $\infty$ , which symbolizes a never ending curve. This symbol spread almost immediately.

But most scientists, as soon as the symbol appeared, considered the argument a paradox. Most the the empiricist philosophers, still belied that we can have no proper idea of the infinite. They believed all our ideas were derived from sense data or impressions, and since all sensory impressions are inherently finite, so too are our thoughts and ideas. Our idea of infinity is merely negative or private.

Modern discussion of the infinite is now regarded as **part of set theory** and mathematics.

## 4. **Infinite Quantities**

4.1. **1. Countably infinite sets.** What is a set? Mathematicians have found that they cannot define this concept and leave it undefined.

It has elements.

Basic notion: $x$ belongs to the set $A$: $x \in A$.

We say that two sets are the "same" if they have the same elements.

Let $A$ and $B$ be sets. If every element of $A$ is also an element of $B$, then we say that $A$ is a subset of $B$: $A \subset B$.

Sets appear in all areas of life. Mainly we deal with **finite sets**, for which we can count the number of their elements. But as soon as we consider natural numbers, or positive rational numbers, we get into the problem of infinity.

**Cantor** (1845-1918) raised the question weather these infinite quantities are comparable. Consider the natural numbers. We can get to those by counting, but it is an infinite procedure. Cantor called such sets **countably infinite sets** and denoted their size (cardinality) by $\aleph_0$ ($\aleph_0$ is the first letter of the Hebrew alphabet). He also noticed the property of $\mathbb{N}$ that there can be a 1-to-1 correspondence between $\mathbb{N}$ and its real subset (like square numbers and $\mathbb{N}$, also by Galilei). (To make the 1-to1 correspondence clear, imagine that you have a seat on an airplane. Exactly one seat is assiged to you, so there is a 1-to-1 correspondence between the passangers and their seats.) Or, even numbers $2n, n \in \mathbb{N}$ form a subset of $\mathbb{N}$, and we can uniquely assign to each $n \in \mathbb{N}$ the even number $2n$.

For infinite sets this a strange phenomena, and is completely against our natural imagination! How can a subset of a set have the same "size" as the set itself? (By same size we mean there is a 1-to-1 correspondence between the two sets.)

This is Cantor's important achievement which shows that infinite sets are beyond our usual "finite" imagination.

**In fact**, infinite sets are exactly those which have a proper subset of the "same size".

The size of even and odd numbers is also $\aleph_0$. Cantor showed:

$$\aleph_0 + \aleph_0 = \aleph_0.$$

**Question:** Is this infinite size enough to cover all existing sets?

**Hilbert Hotel**(Example of **George Gamow**, 1947)

Consider a hypothetical hotel with a countably infinite number of rooms, all of which are occupied. One might be tempted to think that the hotel would not be able to accommodate any newly arriving guests, as would be the case with a finite number of rooms.

*Finitely many new guests*

Suppose a new guest arrives and wishes to be accommodated in the hotel. We can (simultaneously) move the guest currently in room 1 to room 2, the guest currently in room 2 to room 3, and so on, moving every guest from his current room $n$ to room $n+1$. After this, room 1 is empty and the new guest can be moved into that room. By repeating this procedure, it is possible to make room for any finite number of new guests.

*Infinitely many new guests*

It is also possible to accommodate a countably infinite number of new guests: just move the person occupying room 1 to room 2, the guest occupying room 2 to room 4, and, in general, the guest occupying room $n$ to room $2n$ $(2 \times n)$, and all the odd-numbered rooms (which are countably infinite) will be free for the new guests.

*Infinitely many coaches*

It is possible to accommodate countably infinitely many coachloads of countably infinite passengers each. Let us prove the last statement.

1. The guest in the first room should stay, then leave the next room empty.
2. The guest from the second room moves to the third room , and the next 2 rooms room remain empty.
3. The guest from the third room moves to the 6th room, and the next 3 rooms remain empty.
4. The guest of the 4th room moves to the 10th room and the next 4 rooms remain empty.

With this method we can make enough empty rooms for all the guests. Take the guests from the first coach (we can do that, as the number of coaches is countable) and put its guests into the "first" empty rooms. Then we can see that again 1,2,3,... rooms remained empty. Continuing the recursion, we could accommodate infinitely many guests of infinitely many coaches in the Hilbert Hotel.

*Remark.* Here we had to use the "axiom of choice" (later).

Summerizing the above statements we can write:

$$\begin{aligned} \aleph_0 + 1 &= \aleph_0, \\ \aleph_0 + \aleph_0 &= \aleph_0, \\ \aleph_0 \times \aleph_0 &= \aleph_0. \end{aligned}$$

But this example raises a new question: Does there exist a bigger than $\aleph_0$ size set?

4.2. **Uncountable sets.** An infinite set is called uncountable if it is not countable.

Cantor shows that the countable set of any real numbers is not complete. He uses the decimal expression of numbers, Consider the next sequence:

$$\begin{aligned} x_1 &= 1\mathbf{4},13, \\ x_2 &= 1,41\mathbf{4}1, \\ x_3 &= 1,414\mathbf{2}32, \\ x_4 &= 1,41423\mathbf{5}5621, \\ x_5 &= 1,41423562\mathbf{3}5, \\ x_6 &= 1,414235623\mathbf{7} \\ &\vdots \end{aligned}$$

It is clear that $x_1 < x_2 < x_3 < \ldots$ because any two elements of the sequence equal up to a certain decimal point, and because of the first different digits the bold face ones - , any two numbers on the list are different. Now we form a number with the help of the bold face digits. Let it be $x$. It is clear that for a list of arbitrary length we can always find an appropriate $x$ which is the least upper bound of the listed $x_i$-s, and is different from the listed ones:

$$x = 1,\mathbf{414235637}\ldots$$

This way similarly of Euclid's proof, we saw that no such list is complete. As a consequence, a countable list of real numbers is never complete.

**Axel Harnack** (1851-1888) concluded a similar result when he tried to cover the entire real line with finite segments. Let $\{a_1, a_2, a_3, a_4, \ldots\}$ the complete list of all real numbers. Every segment can be covered so that the sum of the length of segments is arbitrary small. Let the total length of the segments $\epsilon$. Then, if we take a segment of length $\epsilon/2$, and we can cover with this $a_1$, then can take the half of the remaining length $\epsilon/4$, and cover with it $a_2$, etc. By cutting to half of the remaining segments we can cover an arbitrary list of real numbers so that the totel length of the segments is $\epsilon$.


Similarly to the previous argument, it is clear that the countably infinite list of the points $a_1$, $a_2$, $a_3$, $a_4, \ldots$ does not cover all the real numbers. Beside, it is also clear that the total length of the segments covering the points $\{a_1, a_2, a_3, a_4, \ldots\}$ can be arbitrary small.

In both presented arguments we notice a similarity which **Cantor** published in **1891**. Consider the subsets of positive integers. Such a subset $H$ can be described as a countably infinite sequence of 0-s and 1-s as follows. The $n$th number is 1, if $H$ contains the number $n$, and is 0 if it does not. Now take $\aleph_0$ such $H$ sets. It means that these sets we can count as $H_1, H_2, H_3, \ldots$, as we can assign to each set a natural number $n$. As before, we have to show that none of such counting contains all the subsets of positive integers. We can do this by constructing a set $H$ which differs in at least one element from the listed $H_i$ sets. Giving such a set is not hard. Put every $n$ into the set $H$ if $H_n$ does not contain $n$, and if it is contained in $H_n$, then $H$ should not contain $n$. Then it is easy to see that $H$ is not in our list , as it differs in $n$ from each $H_n$. Cantor called this kind of reasoning **diagonal**, and it can be discovered in every previous example where we showed the existence of not countable infinite quantities.

Imagine a diagram consisting of countable infinite columns and rows, the rows of which are the sets $H_i$, represented as sequence of 0-s and 1-s, depending on weather $H_i$ contains the positive integer assigned to the column. Then with the help of the sequence in the main diagonal we can construct the set $H$ which definitely does not appear in our list.

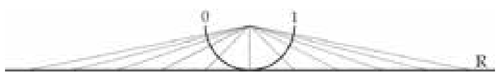
| - | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|
| $H_1$ | **0** | 1 | 1 | 0 | 1 | $\cdots$ |
| $H_2$ | 0 | **0** | 1 | 0 | 1 | $\cdots$ |
| $H_3$ | 1 | 0 | **1** | 1 | 0 | $\cdots$ |
| $H_4$ | 1 | 1 | 1 | **1** | 1 | $\cdots$ |
| $H_5$ | 0 | 1 | 0 | 1 | **0** | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |
| **H** | **1** | **1** | **0** | **0** | **1** | $\cdots$ |

We can see that in constructing the set $H$, we interchanged to the opposite each element of the main diagonal, from which we can conclude the size of the set which consists of all subsets of positive integers. We can determine the size of this set, if we consider the process of constructing the set $H$. $H$ is a set of sequences consisting of 0-s and 1-s, where to any place we can write two values , following the above rule. When we try to construct all such 0, 1 sequences, we write in $\aleph_0$ spaces one value out of 2. This means $\aleph_0$ product terms by the following:

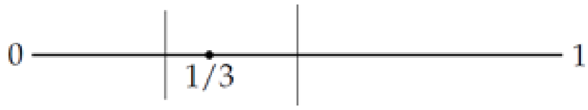
$$2 \times 2 \times 2 \times 2 \ldots$$

This way we were able to define with the **diagonal argument** $2^{\aleph_0}$ different sets of positive integers. On the other hand, Cantor's reasoning also shows that $\aleph_0 < 2^{\aleph_0}$, because there are 1 to 1 correspondences between the natural numbers and some of its subsets, but not all of its subsets. We have seen in our previous examples that there exists such a set, the size of which is bigger than $\aleph_0$. Question: For the size of such sets is the $2^{\aleph_0}$, obtained by the diagonal argument satisfactory?

First, consider the number of points on the line. We have seen that this is definitely bigger than $\aleph_0$. With the help of the next figure we prove geometrically that there can be obtained a 1 to 1 correspondence between the points of the segment $(0, 1)$ and the whole number line. To do this, we just "bend" the $(0, 1)$ segment into a half circle. Then the radii from the center of the half circle to the points of half circle give an appropriate correspondence between $(0, 1)$ and the real line.




To identify real numbers let us use the following coding method. Define the binary extension of an $x$ real number in the interval $(0, 1)$ like this: let us first cut into half the interval. If $x$ is in the left side half interval, let us write after the binary point 0, if it lies in the right side, let us write 1. After, to define the second binary digit, repeat the process for the half interval containing $x$. Example: we get the binary extension of 1/3 as follows:




$$.01010101010101010101010101010101010101\ldots$$

A problem might arise if the point $x$ lies on the dividing line, like $x = 1/2$ or $x = 1/4$. Then we put $x$ to one of the sides. If for instance $x = 1/2$, then putting $x$ to the left side, we get the following extension:

$$.0111111111111111111111111111111111111\ldots$$

If we put $x$ to the right side, we get

$$.10000000000000000000000000000000000\ldots$$

We can overcome this problem and we get a 1 to 1 correspondence between the set $\mathbb{R}$ and the set of all subsets of natural numbers.

## 5. **New problems**

The diagonal reasoning can be applied to every set, so one can show the every set has more subsets then elements. From this it follows that we can not find a biggest set. So it does not exist the set of all sets, because if it would exist, it would be the biggest set. Along this reasoning the question arises: What do we mean by “set”, if the set of all sets does not exist. Cantor could not precisely answer this question, he rather restricted his studies to sets coming from well-defined operations, like counting the subsets of a set.

The question was more complicated for the philosophic mathematicians, like **Gottlob Frege** (1848-1925), and **Bertrand Russell** (1872-1970). They thought that every property $P$ uniquely defines a set , exactly the one, all the elements of which satify property $P$. The problem arises if we define property $P$ as a set, is it that, when we get to the problem of a set containing all the sets. Russell gave the next popular barber example for this paradox: Once the barber of a town stated that he will cut the hair for every man who does not cut his hair himself. But if this is true, who will cut the hair of the barber?

To overcome the problem, Russell made the following restriction. Consider the set of all such sets, which do not contain themselves as an element. So if we move the barber from the town, we resolve the paradox. Russel’s reasoning convinced the mathematical society about the necessity of defining certain basic notions in the entire mathematics.

At the beginning of the 20th century, **Ernst Zermelo** (1871-1953) thought that set theory can be well-founded by introducing axioms. This way he formalized Cantor's intuition, that every set can be constructed with well-defined operations from a set, like from the set of natural numbers.

## 6. **Axiomatic approach of infinity**

6.1. **Set theoretical axioms.** Most of the axioms used in today's set theory originates from Zermelo,

and with an important addition from **Abraham Fraenkel** (1891-1965), and we call it **Zermelo-Fraenkel system of axioms** (1904 and 1922).

**1. Axiom of Extensionality**: If two sets have the same elements, the two sets are equal.

**2. Axiom of Separation** (or **Axiom of Subsets**): If $A$ is a set and $P$ is a condition on elements of $A$ then those elements of $A$ which satisfy condition $P$ form a set.

**3. Axiom of Pair** (or **Axiom of Unordered Pairs**): If $A$ and $B$ are sets, then there exists a set consisting of exactly $A$ and $B$.

**4. Axiom of Union** (or **Axiom of Sum Set**): For any set of sets $\mathcal{F}$, there is a set $A$, containing every element that is an element of some member of $\mathcal{F}$.

**5. Axiom of Power Set**: For every set $A$ there exists its power set $\mathcal{P}(A)$, which consists of all subsets of $A$.

**6. Axiom of Infinity**: An inductive infinite set exists. Its elements are: $\emptyset$, $\{\emptyset\}$, $\{\emptyset,\{\emptyset\}\}$, $\{\emptyset, \{\emptyset\},\{\emptyset,\{\emptyset\}\}\}$, ...

**7. Axiom of Replacement**: The image of a set under any definable function is also a set.

**8. Axiom of Regularity** or (**Axiom of Foundation**) Every nonempty set $A$ contains an element that is disjoint from $A$.

**9. Axiom of Choice**: For every set $S$ of nonempty disjoint sets, there exists a (choice) function $f$ defined on $S$ such that for each set of $S$, $f(x) \in x$.

*Remark.* We have used these axioms, including the axiom of choice, in our previous examples, like the union of countable many countable sets is countable.

Let us denote the elements of the sets $A_0, A_1, A_2, \dots$ by

$$\begin{aligned} A_0 &= \{a_{00}, a_{01}, \dots\} \\ A_1 &= \{a_{10}, a_{11}, \dots\} \\ A_2 &= \{a_{20}, a_{21}, \dots\} \\ A_3 &= \{a_{30}, a_{31}, \dots\} \\ &\vdots \end{aligned}$$

Let us take the union of these sets:

$$X = A_0 \cup A_1 \cup A_2 \cup A_3 \ldots = \{a_{00}, a_{01}, a_{10}, a_{20}, a_{11}, a_{02}, a_{03}, a_{12} \ldots\}$$

Choose an enumeration of elements of $A_i$ (axiom of choice!)

$$\begin{array}{c|cccccccc}
A_0 & a_{00} & \rightarrow & a_{01} & & a_{02} & \rightarrow & a_{03} & \cdots \\
 & & \swarrow & & \nearrow & & \swarrow & & \\
A_1 & a_{10} & & a_{11} & & a_{12} & \cdots & & \\
 & \downarrow & \nearrow & & \swarrow & & & & \\
A_2 & a_{20} & & a_{21} & \cdots & & & & \\
 & & \swarrow & & & & & & \\
A_3 & a_{30} & \cdots & & & & & & \\
\vdots & \downarrow & & & & & & &
\end{array}$$

Let us illustrate the advantage of the axiomatic set theory on an example. **János Neumann** (1903-1957) defined in 1920 the natural numbers as follows. Let $\emptyset$ be the empty set. $1 = \{\emptyset\}, 2 = \{\emptyset \{\emptyset\}\}, 3 = \{\emptyset, 1, 2\}, ...$ Notice that $n + 1 = n \cup \{\emptyset\}$ and $m < n$ if and only if $m$ is a subset of $n$. With this method we constructed the function of "inheritance", which is an unavoidable tool of many mathematical proofs, and we also obtained the definition of the relation "less than", which means we can start to built the arithmetic. We have the set $\{1, 2, 3, 4, \ldots\}$ of natural numbers, we can built a set of size $2^{\aleph_0}$, and many other things in today's mathematics. An important result is the relation $<$, with which we can compare elements of natural numbers, and we can put them in a sequence.

From this comes a question: Which sets have an ordering relation , and what conditions such a comparison has to satisfy.

6.2. **Order and ordering.** The relation $<$ used for natural numbers have to satisfy the next three conditions: **-irreflexive**: $x < x$ is not satisfied for any element $x$ of the set $A$ - **transitive**: if $x,\ y,\ z$ are elements of $A$ and $x < y, y < z$ then $x < z$; - **trichotom**: for $x,\ y$ from $A,\ x < y,\ x = y$ or $y < x$.

Assume we have two ordered sets $A$ and $B$. We call a function $A \rightarrow B$ order preserving if for $x < y,\ f(x) < f(y)$. If such a function is isomorphic we say that the ordered sets $A$ and $B$ are isomorphic. The joint property of isomorphic ordered sets we call order type. Let us check which of those ordered sets have a minimal element. (So far we only see the finite sets, the set of natural numbers and its subsets). Call an ordered set $A$ **well-ordered**, if any subset of $A$ has a least element.

We call the order type of well-ordered set **ordinal number** or **ordinal**.

**Question: Can we compare two ordinals?** Yes, one can define an ordering between ordinals, so in some sense, ordinals are generalizations of natural numbers. We call a subset $B$ of an ordered set $A$ **prefix**, if for $x, y \in A$ such that $y < x$, $y \in B$.

**Example**. A prefix of $\mathbb{N}$ is $\{1, 2, 3, ..., 96\}$ and of course also the two trivial ones: 0 and $\mathbb{N}$.

Let us introduce the **prefix defined by an element** $a$. This is the subset $B := x \in A : x < a$. With this we are able to compare two ordinals. Let $\alpha$ and $\beta$ be two ordinals. We say that $\alpha < \beta$, if the ordinal of the well-ordered set $A$ is $\alpha$, the ordinal of the well-ordered set $B$ is $\beta$, and $A$ is isomorphic to a prefix defined by an element of $B$. It can be shown that this comparison is irreflexive, transitiv and trichotom.

Now if we have a well-ordered set $A$ and its subset $B$ with the same well-ordering, then for the ordinal $\beta$ of $B$, $\beta < \alpha$.

After this, we can continue in the other direction and can enlarge our given sets, approaching this way infinity. As an example, take a well-ordered set $A$ with ordinal $\alpha$, and add to it an element $z$ which is greater than any element of A. The new set $B = A \cup z$ is well-ordered with the ordering in $A$, as $z$ is greater than any element in $A$. The ordinal of $B$ only depends on $\alpha$. Let the constructed ordinal of $B$ be $\alpha + 1$. Call ordinals of this type **subsequent ordinals**. Those nonzero ordinals which are not of subsequent type, call **limit ordinals**. This is equivalent to the following. If we have a well-ordered set $A$, then we can talk about subsequent ordinal if $A$ has a largest element. If there is non, then we talk about limit ordinal.

As for natural numbers, for ordinals we can also define operations.

**1. Sum of ordinals**

Let $A$ and $B$ disjoint well-ordered sets. Let $A$ and $B$ keep their original ordering. We require that every element of $A$ is before the elements of $B$.

Example: Let $A$ be the set of odd natural numbers, $B$ the set of even natural numbers. Their union is $N$, which can be well-ordered. Consider now the following ordering: say "$m$ is less than $n$" if $m$ is odd and $n$ even, or both numbers have the same parity. This way we gave the following ordering:

1, 3, 5, ... , 2, 4, 6, ...

This construction can be naturally generalized to any, pairwise disjoint system of sets. The sum operation is associative, but not necessarily commutative.

### 2. Ordered product of ordinals

We define the product of the well-ordered sets $A$ and $B$ as adding to $A$ $B$ times itself. For every $b \in B$, we introduce the set of ordered pairs $(a, b)$ where $a \in A$. The **ordering** we define as follows.

If $(a, b), (c, d) \in A \times B$ and $(a, b) < (c, d)$, then $b < d$ or $b = d$ or $a < c$.

Warning: It is important from which side we multiply. If from the left, like $2\omega$, we can get a set similar to $\omega$. But if we consider $1\omega + 1\omega$, this will be different from $\omega$.

### 3. Power of ordinals

Use the product of ordinals.

$$\alpha^0 := 1$$

$$\alpha^{\beta+1} = \alpha^\beta \times \alpha,$$

where $\alpha^\beta := \{\sup \alpha^\gamma, \gamma < \beta\}$. With these operations let us try to enlarge ordinals.

$$\omega, \omega + 1, \omega + 2, \omega + 3, ..., \omega \times 2,$$

where $\omega \times 2$ is the supremum of the previous ones.

Now continue as follows:

$$\omega \times 2 + 1, \omega \times 2 + 2, \omega \times 2 + 3, ...$$

The least upper bound of these denote by $\omega \times 3$.

This initiates a new sequence of growing ordinals, the least upper bound of which is $\omega \times 4$, etc. Consider now the next sequence of the so far obtained ordinals:

$$\omega, \omega \times 2, \omega \times 3, ...$$

the supremum of which being $\omega \times \omega = \omega^2$. This can be again exceeded:

$$\omega^2 + 1, \omega^2 + 2, \omega^2 + 3, ..., \omega^2 + \omega,$$

which again can be enlarged:

$$\omega^2 + \omega \times 2, \omega^2 + \omega \times 3, \omega^2 \times 4, ..., \omega^2 + \omega^2 = \omega^2 \times 2$$

This is followed in a natural way by the sequence of bigger ordinals $\omega^3, \omega^4, \omega^5, ...$ and their supremum $\omega^\omega$.

Let me make this process more spectacular. Let us try to imagine the ordinals as the set of rational numbers in $(0, 1)$, with the usual ordering $<$. Let $\omega$ be the ordinal of the set $\{1/2, 3/4, 7/8, \dots\}$ and let us represent every element of this set by a vertical line segment, the length of which approaching 0, as we approach $\omega$.

If we can represent this way the ordinal $\alpha$, than also $\alpha+1$. First place the fractions representing $\alpha$ into $(0, 1/2)$ by dividing every element by 2.

### 3. Power of ordinals

Use the product of ordinals.

$$\alpha^0 := 1$$

$$\alpha^{\beta+1} = \alpha^\beta \times \alpha,$$

where $\alpha^\beta := \{\sup \alpha^\gamma, \gamma < \beta\}$. With these operations let us try to enlarge ordinals.

$$\omega, \omega + 1, \omega + 2, \omega + 3, ..., \omega \times 2,$$

where $\omega \times 2$ is the supremum of the previous ones.

Now continue as follows:

$$\omega \times 2 + 1, \omega \times 2 + 2, \omega \times 2 + 3, ...$$

The least upper bound of these denote by $\omega \times 3$.

This initiates a new sequence of growing ordinals, the least upper bound of which is $\omega \times 4$, etc. Consider now the next sequence of the so far obtained ordinals:

$$\omega, \omega \times 2, \omega \times 3, ...$$

the supremum of which being $\omega \times \omega = \omega^2$. This can be again exceeded:

$$\omega^2 + 1, \omega^2 + 2, \omega^2 + 3, ..., \omega^2 + \omega,$$

which again can be enlarged:

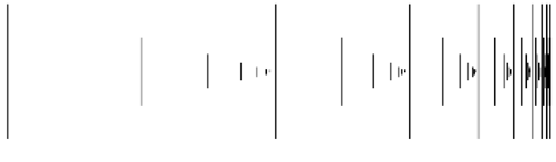
$$\omega^2 + \omega \times 2, \omega^2 + \omega \times 3, \omega^2 \times 4, ..., \omega^2 + \omega^2 = \omega^2 \times 2$$

This is followed in a natural way by the sequence of bigger ordinals $\omega^3, \omega^4, \omega^5, ...$ and their supremum $\omega^\omega$.

Let me make this process more spectacular. Let us try to imagine the ordinals as the set of rational numbers in $(0, 1)$, with the usual ordering $<$. Let $\omega$ be the ordinal of the set $\{1/2, 3/4, 7/8, \dots\}$ and let us represent every element of this set by a vertical line segment, the length of which approaching 0, as we approach $\omega$.

If we can represent this way the ordinal $\alpha$, than also $\alpha+1$. First place the fractions representing $\alpha$ into $(0, 1/2)$ by dividing every element by 2.

Then add to this system the point $1/2$, as the representative of $\alpha+1$. If $\alpha_1 < \alpha_2 < \alpha_3 < ...$ is representable, then so is its supremum. For this place $\alpha_1$ into $(0, 1/2)$, $\alpha_2$ into $(1/2, 3/4)$, $\alpha_3$ into $(3/4, 7/8)$, .... Finally put into its place the supremum of the constructed $\alpha_1, \alpha_2, ...$ system, the 1. Thus way we can even represent the ordinal $\omega^\omega$ with the appropriate set of rational numbers:



6.3. **Continuum hypothesis. Question:** *Can we get further in the notion of infinity?*

So far we have seen several examples of incomprehensibility of ordinals, but it is imprtant to notice that they all were countable or countably infinite. Can we get further with this procedure?

We can also define the cardinality of ordinals, as there is an order-preserving bijection between sets of given ordinal.

**Theorem. There exists an uncountable ordinal.**

With this we can overcome the countable ordinals. Let us take the smallest among them, and denote it by $\omega_1$, its cardinality by $\aleph_1$. This way we constructed the smallest ordinal which is greater than $\aleph_0$, but - as we have seen before - , there exists a cardinal which is bigger than $\aleph_1$. But in this case we can choose among those the smallest one, $\omega_2$, the cardinality of which we call $\aleph_2$. With this method we can get the recursion of transfinite ordinals.

**Question**: Where is the continuum cardinality in this list? It would be good if the uncountable set obtained by the two methods, the $2^{\aleph_0}$ subsets of $\mathbb{N}$ and the newly obtained $\aleph_1$, would represent the same cardinality:

$$2^{\aleph_0} = \aleph_1.$$

This problem was already noticed by Cantor in 1883. He called it "continuum hypothesis", but the decision of its truth goes beyond the human power.

In the 20th century there were two important breakthroughs in understanding what can be proved in a given system and what not. In 1940

**Kurt G´odel** (1906-1970) was able to prove that the continuum hypethesis can not be denied on the basis of the given axiom system. In 1963 **Paul Cohen** (1934-2007) showed the independence of the continuum hypothesis. He gave a method which makes it possible to construct infinitely many models of set theory. Cohen himself thought there might be many cardinals between countable and continuum, but so far neither the validity nor the impossibility can be justified.

**Question: Does a Universe exist without contradictions ?**

For interested readers, here are two **general references**.

1. http://en.wikipedia.org/wiki/Infinity
2. http://en.wikipedia.org/wiki/Godel, Escher, Bach

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